Multivariate Statistics Applied to Assess Measurement Uncertainty of Complex Reflection Coefficient

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***Abstract*—In this paper we show an alternative mathematical interpretation of the error propagation law for complex quantities established in supplement 2 of the GUM [1].**
We use this interpretation to study VNA's one port reflection measurement which includes several terms that represent complex quantities. We show two different approaches to solve the problem that arises when trying to establish the variance matrix of the sum of two complex quantities. We also give explicit formulae to estimate uncertainty with both approaches.

Keywords*—*GUM, multivariate statistics, RF metrology, VNA measurement, complex quantities, measurement uncertainty.

I. Theory

Supplement 2 of the GUM [1] establishes a method for the expression of uncertainty in multiple output measurements. This means determining the variance matrix of the output vector. In the special case where measurand and input quantities are complex, an alternative formulation of the error propagation law is presented in [2]. This formulation is a matrix analogy of the well known law in the univariate case.
Let $f : \mathbb{C}^m \to \mathbb{C}$ be an analytic function and the complex measurand $Y = f(z_1, \dots, z_m)$, [2] shows that the variance matrix of Y is

$$\mathbf{var}\,[Y] = \sum_{i=1}^{m}\sum_{j=1}^{m} J(c_i)\mathbf{cov}\,[z_i, z_j]\, J(c_j)^t \tag{1}$$

Where $c_i = \frac{\partial f}{\partial z_i}$ are sensitivity coefficients, $J(c_i)$ their matrix representation and $J(c_j)^t$ the transpose of $J(c_j)$

$$J(c_i) = \begin{pmatrix} \mathrm{Re}(c_i) & -\mathrm{Im}(c_i) \\ \mathrm{Im}(c_i) & \mathrm{Re}(c_i) \end{pmatrix} \tag{2}$$

The covariance matrix of two complex variables is

$$\mathbf{cov}\,[z_i, z_j] = \begin{pmatrix} u[\mathrm{Re}(z_i),\mathrm{Re}(z_j)] & u[\mathrm{Re}(z_i),\mathrm{Im}(z_j)] \\ u[\mathrm{Im}(z_i),\mathrm{Re}(z_j)] & u[\mathrm{Im}(z_i),\mathrm{Im}(z_j)] \end{pmatrix} \tag{3}$$

Where $u[\mathrm{Re}(z_i), \mathrm{Re}(z_j)]$ is the covariance between $\mathrm{Re}(z_i)$ and $\mathrm{Re}(z_j)$. $u[\mathrm{Re}(z_i), \mathrm{Re}(z_i)] = u^2[\mathrm{Re}(z_i)]$ is the variance of $\mathrm{Re}(z_i)$.
If z and w are complex variables and c and d are constant complex values, covariance matrices have the following properties

$$\mathbf{var}\,[z] = \mathbf{cov}\,[z, z] \tag{4}$$
$$\mathbf{cov}\,[z, w] = \mathbf{cov}\,[w, z]^t \tag{5}$$
$$\mathbf{cov}\,[cz, dw] = J(c)\mathbf{cov}\,[z, w]\, J(d)^t \tag{6}$$
$$\mathbf{cov}\,[z_1 + z_2, w] = \mathbf{cov}\,[z_1, w] + \mathbf{cov}\,[z_2, w] \tag{7}$$
$$\mathbf{var}\,[z + w] = \mathbf{var}\,[z] + \mathbf{var}\,[w] + \mathbf{cov}\,[z, w] + \mathbf{cov}\,[w, z] \tag{8}$$
$$\mathbf{var}\,[-z] = \mathbf{var}\,[z] \tag{9}$$

Using (4), (6) and (8) in equation (1) the variance matrix of the measurand can be interpreted as

$$\mathbf{var}\,[Y] = \sum_{i=1}^{m}\sum_{j=1}^{m} \mathbf{cov}\,[c_i z_i, c_j z_j] = \mathbf{var}\left[\sum_{i=1}^{m} c_i z_i\right] \tag{10}$$

The complex variables z and w are uncorrelated if

$$\mathbf{cov}\,[z, w] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{11}$$

From (8) we get that

$$\mathbf{var}\,[z + w] = \mathbf{var}\,[z] + \mathbf{var}\,[w] \tag{12}$$

We say z is "circular" and has a circular variance matrix if

$$\mathbf{var}\,[z] = u_z^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{13}$$

Where $u_z^2 = u^2[\mathrm{Re}(z)] = u^2[\mathrm{Im}(z)]$. If c is a constant complex value,

$$\mathbf{var}\,[cz] = |c|^2 u_z^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{14}$$

II. Measurement Model

The following equation expresses the true reflection coefficient Γ of a one port device measured with a VNA.

$$\Gamma = \frac{\Gamma_m - D}{M(\Gamma_m - D) + T} - R \tag{15}$$

Here Γ_m is the measured value of Γ. D, M and T are the residual directivity, source-match and reflection tracking respectively. R accounts for external error terms, such as cable flexibility, stability, etc. All terms are complex quantities. The following approximations can be considered: $T \approx 1$, $M, D, R \approx 0$ and $\Gamma_m \approx \Gamma$. After complex differentiation and using the approximate values,

$$\frac{\partial \Gamma}{\partial D} = -1 \quad \frac{\partial \Gamma}{\partial T} = -\Gamma \quad \frac{\partial \Gamma}{\partial M} = -\Gamma^2 \quad \frac{\partial \Gamma}{\partial R} = -1 \tag{16}$$

Replacing in (10) and using (9), we have that

$$\mathbf{var}\,[\Gamma] = \mathbf{var}\left[D + \Gamma^2 M + \Gamma T + \Gamma_m + R\right] \tag{17}$$

III. Uncertainty Analysis

All input variables are assumed uncorrelated , and except for Γ_m they all satisfy (13).
The Ripple technique described in [3] is a standard approach to assess $\mathbf{var}\,[D]$ and $\mathbf{var}\,[M_{ef}]$. Where M_{ef}, shown in Fig.1, is the "Effective Test Port Match ". This quantity is similar to M and is assumed to satisfy (13). Guidelines in [3] establish

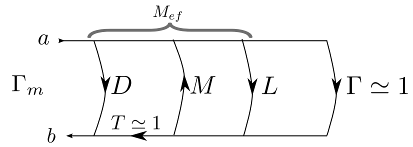

Fig. 1. Ripple signal flow graph for M_{ef}

to use M_{ef} as M, and warns that M_{ef} could be correlated with D. Using (12) and (14) in (17) we get

$$\mathbf{var}[\Gamma] = \mathbf{var}\left[D + \Gamma^2 M_{ef}\right] + |\Gamma|^2 \mathbf{var}[T] + \mathbf{var}[\Gamma_m] + \mathbf{var}[R] \tag{18}$$

The last three terms of the sum do not present any analytical difficulties. For the first one, [3] neither explains how to calculate the covariance needed nor gives a bivariate treatment to the problem.

In order to overcome this situation we present two different approaches we developed. One that overestimates $\mathbf{var}\left[D + \Gamma^2 M_{ef}\right]$ and another where the covariance between M_{ef} and D is calculated. Although the second approach is more rigorous, it makes more assumptions about the model.

The first approach is also of general interest beyond the case of the reflection coefficient measurement. It serves as an example of what can be done to establish the variance matrix of the sum of two complex quantities when their covariance matrix is unknown.

A. Overestimation of the Variance of the Sum

Using (8), the fact that D and M_{ef} satisfy (13), and assuming that $u[\mathrm{Re}(D), \mathrm{Im}(M_{ef})] = u[\mathrm{Im}(D), \mathrm{Re}(M_{ef})] = 0$

$$\mathbf{var}\left[D+\Gamma^2 M_{ef}\right] = \begin{pmatrix} u^2[\mathrm{Re}(D)+\mathrm{Re}(\Gamma^2 M_{ef})] & 0 \\ 0 & u^2[\mathrm{Im}(D)+\mathrm{Im}(\Gamma^2 M_{ef})] \end{pmatrix} \tag{19}$$

Due to missing information about covariance, diagonal elements can not be calculated. Cauchy-Schwarz inequality for random scalars is used to bound them in order to get a worst case variance matrix as follows

$$\mathbf{var}\left[D+\Gamma^2 M_{ef}\right] = (u_D + |\Gamma|^2 u_{M_{ef}})^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{20}$$

B. Covariance Matrix of D and M_{ef}

Solving the flow graph in Fig. 1 yields

$$M_{ef} \approx M + D + L \tag{21}$$

Where L is the reflection coefficient of the air-line used for the Ripple technique. L is assumed to satisfy (13) and is uncorrelated with all input variables. Using (4), (7) and (21)

$$\begin{aligned} \mathbf{cov}[M_{ef}, D] &= \mathbf{cov}[M, D] + \mathbf{cov}[D, D] + \mathbf{cov}[L, D] \\ &= \mathbf{var}[D] \end{aligned} \tag{22}$$

Using (5), (6), (8), (14), (22) and assuming that D is circular, it is possible to obtain the following equality

$$\begin{aligned} \mathbf{var}\left[D + \Gamma^2 M_{ef}\right] = & \mathbf{var}[D] + |\Gamma|^4 \mathbf{var}[M_{ef}] + \ldots \\ & \ldots + (J(\Gamma^2) + J(\Gamma^2)^t)\mathbf{var}[D] \end{aligned} \tag{23}$$

Finally, this can be expressed as

$$\mathbf{var}\left[D + \Gamma^2 M_{ef}\right] = \left((1 + 2\mathrm{Re}(\Gamma^2))u_D^2 + |\Gamma|^4 u_{M_{ef}}^2\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{24}$$

IV. Measurand's Variance Matrix

In this section we establish $\mathbf{var}[\Gamma]$ for both approaches. In order to get a circular variance matrix for the measurand we overestimate $\mathbf{var}[\Gamma_m]$ with a circular variance matrix. A common practice is to do this with

$$u_{\Gamma_m}^2 = u^2(\mathrm{Re}(\Gamma_m)) + u^2(\mathrm{Im}(\Gamma_m)) \tag{25}$$

For the first approach, using (20) in equation (18) we get an overestimated variance matrix

$$\mathbf{var}[\Gamma] = u_{c1}^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{26}$$

where

$$u_{c1}^2 = (u_D + |\Gamma|^2 u_{M_{ef}})^2 + |\Gamma|^2 u_T^2 + u_{\Gamma_m}^2 + u_R^2 \tag{27}$$

For the second approach, using (24) in equation (18) we get

$$\mathbf{var}[\Gamma] = u_{c2}^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{28}$$

where

$$u_{c2}^2 = (1+2\mathrm{Re}(\Gamma^2))u_D^2 + |\Gamma|^4 u_{M_{ef}}^2 + |\Gamma|^2 u_T^2 + u_{\Gamma_m}^2 + u_R^2 \tag{29}$$

In order to compare both matrices it is enough to compare u_{c1} and u_{c2}.

V. Comparision of both approaches

We show u_{c1} and u_{c2} defined in (27) and (29) for three real measurement of low, medium and high reflection coefficients at 18 GHz with a type N connector

	Γ_{low}	Γ_{med}	Γ_{high}
$\lvert\Gamma\rvert$	0.058	0.559	0.980

The values obtained for u_{c1} and u_{c2} were

$\times 10^{-3}$	Γ_{low}	Γ_{med}	Γ_{high}
u_{c1}	7.8	11.9	20.5
u_{c2}	7.8	7.8	12.2

VI. Conclusion

Formulating the variance matrix of a measurand as in (10) allows a better understanding of the error propagation law. The approach shown in (III-A) is a useful resource when the covariance between two complex variables is unknown. In (V) we showed that overestimation was too big for measurements with medium or high reflection values. Therefore, overestimation should be avoided whenever possible, for example using an approach similar to (III-B).

Equations (28) and (29) are of special interest for one port reflection measurements as they establish an explicit expression for the measurand's variance matrix.

References

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